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Traveling wavefront in a Hematopoiesis model with time delay[☆]

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ABSTRACT

This paper is concerned with a reaction–diffusion equation with time delay, which describes the dynamics of the blood cell production. The existence of the traveling wavefront is given by using the upper–lower solution technique and the monotone iteration.

1. Introduction

In this paper, we consider the following reaction–diffusion equation with time delay:

$$\frac{\partial N(t,x)}{\partial t} - \frac{\partial^2 N(t,x)}{\partial x^2} = -\delta N(t,x) - \frac{\beta\theta^n N(t,x)}{\theta^n + N^n(t,x)} + \frac{2\beta\theta^n N(t-\tau,x)}{\theta^n + N^n(t-\tau,x)} e^{-\gamma\tau}, \tag{1.1}$$

where diffusion is introduced to describe the spatial movement of substances from high to low concentration. The corresponding ordinary differential equation of (1.1) is

$$\frac{\partial N(t,x)}{\partial t} = -\delta N(t,x) - \frac{\beta\theta^n N(t,x)}{\theta^n + N^n(t,x)} + \frac{2\beta\theta^n N(t-\tau,x)}{\theta^n + N^n(t-\tau,x)} e^{-\gamma\tau}, \tag{1.2}$$

which was first proposed by Mackey [1] to study the dynamics of blood cell production. $N(t)$ denotes the density of mature stem cells in blood circulation, τ is the time delay between the production of immature stem cells in the bone marrow and their maturation for release in the circulating blood stream, $\delta, \beta, \theta, \gamma \in (0, +\infty)$ and $n \in (1, +\infty)$ are positive constants which represent some specific meanings in blood circulation. For instance, δ is the lost rate of the cells from the circulation. In this model, the flux $\frac{2\beta\theta^n N(t-\tau,x)e^{-\gamma\tau}}{\theta^n + N^n(t-\tau,x)}$ of the cells into the circulation from the stem cell compartment depends on the number of cells $N(t-\tau,x)$ at time $t-\tau$. For more details about Hematopoiesis model we refer the readers to the articles of Mackey [1,2] and the references cited therein.

Weng and Dai [3] considered Eq. (1.2) and proved that the positive equilibrium could be globally attractive under some assumptions. Recently, Wang [4] investigated the following general equation

$$\frac{\partial u(t,x)}{\partial t} - \Delta u(t,x) = -\delta u(t,x) + f(u(t-\tau,x)) \tag{1.3}$$

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with the Neumann boundary condition and some initial condition. The oscillatory behavior of solutions about the positive equilibrium of (1.3) was obtained. They gave sufficient and necessary conditions for global attractivity of the zero solution. Moreover, they applied these results to another diffusive Hematopoiesis models as following:

$$\frac{\partial N(t,x)}{\partial t} - \Delta N(t,x) = -\delta N(t,x) + \frac{\beta\theta^n N(t-\tau,x)}{\theta^n + N^n(t-\tau,x)}. \tag{1.4}$$

Global attractivity of the positive equilibrium of (1.3) without diffusion term has been investigated by Gopalsamy and Kulenvic [5].

By the transition of variable $N(t,x) = \theta u(t,x)$ we may rewrite (1.1) in the equivalent form

$$\frac{\partial u(t,x)}{\partial t} - \frac{\partial^2 u(t,x)}{\partial x^2} = -\delta u(t,x) - \frac{\beta u(t,x)}{1+u^n(t,x)} + \frac{2\beta u(t-\tau,x)}{1+u^n(t-\tau,x)} e^{-\gamma\tau}. \tag{1.5}$$

Our aim is to consider the existence of traveling wavefront of the Eq. (1.5).

As we know, the theory of traveling wave solutions of reaction–diffusion equations is one of the fast developing area of modern mathematics and has attracted much attention due to its significance in biology, chemistry, epidemiology and physics. The traveling wave problems for reaction–diffusion systems without delay have been studied by many authors, see [6,7]. Moreover, the research on traveling wave solutions to reaction–diffusion systems with delay have been widely studied in the literature [8–13]. Schaaf [12] systematically studied two scalar reaction–diffusion equations with a single discrete delay by using the phase plane technique, the maximum principle for parabolic functional differential equations and the general theory of ordinary differential equations. The monotone iteration technique has been used in [14] and the degree theory has been adopted in [8,10].

The organization of this paper is as follows. In Section 2, we will introduce the technique developed by Wu and Zou [14]. Then we give the condition for establishing the positive equilibriums and obtain the existence of traveling wavefronts in Section 3.

2. Preliminaries

To investigate the existence of traveling wavefront of (1.5), the approach we used is based on the construction of a suitable super and lower solutions. For convenience, here we state the idea which is inspired from [14].

Consider a scalar reaction–diffusion equation with time delay:

$$\frac{\partial u(t,x)}{\partial t} = D\frac{\partial u^2(t,x)}{\partial x^2} + f(u_t(x)), \tag{2.1}$$

where $t \geq 0, x \in \mathbb{R}, u \in \mathbb{R}$, the positive constant D is diffusion coefficient. The function $f : C([-\tau,0],\mathbb{R}) \to \mathbb{R}$ is continuous and $u_t(x)$ is an element in $C([-\tau,0],\mathbb{R})$ parameterized by $x \in \mathbb{R}$ and given by

$$u_t(x)(s) = u(t+s,x), \quad s \in [-\tau,0],\ t \geqslant 0,\ x \in \mathbb{R}.$$

Looking for traveling wave solutions of the form $u(t,x) = \phi(x+ct)$, letting $s = x+ct$ and replacing s by t lead to a second-order functional differential equation

$$D\phi''(t) - c\phi'(t) + f_c(\phi_t) = 0, \quad t \in \mathbb{R}, \tag{2.2}$$

where $f_c : X_c \triangleq C([-c\tau,0],\mathbb{R}) \to \mathbb{R}$ is define by

$$f_c(\psi) = f(\psi^c), \qquad \psi^c(s) = \psi(cs), \quad s \in [-\tau,0].$$

Now we assume that

(A1) There exists $K > 0$ such that $f_c(\hat{0}) = f_c(\hat{K}) = 0$ and $f_c(\hat{u}) \neq 0$ for $u \in (0,K)$, where $\hat{u}$ denotes constant function taking the value u on $[-c\tau,0]$.

(A2) There exists $\alpha \geqslant 0$ such that

$$f_c(\phi) - f_c(\psi) + \alpha[\phi(0) - \psi(0)] \geqslant 0$$

for $\phi, \psi \in X_c$ with $0 \leqslant \psi(s) \leqslant \phi(s) \leqslant K,\ s \in [-c\tau,0]$.

If for some $c > 0$, (2.2) has a monotone solution ϕ defined on $\mathbb{R}$ such that

$$\lim_{t\to-\infty}\phi(t) = 0, \qquad \lim_{t\to\infty}\phi(t) = K, \tag{2.3}$$

then $u(t,x) = \phi(x+ct)$ is called a traveling wavefront of (2.1) with speed c.

Define the profile set for traveling wavefronts of (1.5) by

$$\Gamma = \left\{\phi \in C(\mathbb{R},\mathbb{R}^n) : \begin{array}{l}\text{(i) } \phi \text{ is nondecreasing in } \mathbb{R},\\ \text{(ii) } \lim\limits_{t\to-\infty}\phi(t) = 0, \quad \lim\limits_{t\to\infty}\phi(t) = K,\end{array}\right\}.$$

The upper and lower solution for (2.1) are defined as follows:

Definition 2.1. The piecewise smooth functions $\overline{\phi}$ and $\underline{\phi}$ in $C(\mathbb{R},\mathbb{R})$ are called upper and lower solutions of (2.1) if $\overline{\phi} \geqslant \underline{\phi}$ and if

$$c\overline{\phi}'(t) \geqslant D\overline{\phi}''(t) + f_c(\overline{\phi}_t), \quad t \in \mathbb{R}$$

and $\underline{\phi}$ satisfies the above differential inequalities in reversed order.

Now we present a scalar version of [14] (Theorem 3.6) and [10] (Theorem 2.2).

Theorem 2.1. *Assume that* (A1) *and* (A2) *hold, if* (2.2) *has an upper solution* $\overline{\phi}$ *in* Γ *and a lower solution* $\underline{\phi}$ *(which is not necessarily in* Γ*) with* $0 \leqslant \underline{\phi}(t) \leqslant \overline{\phi}(t) \leqslant K$, $\underline{\phi}(t) \not\equiv 0$ *in* $\mathbb{R}$, *and* $\tilde{\varphi}'(t+) \leq \tilde{\varphi}'(t-)$, $\hat{\varphi}'(t+) \geq \hat{\varphi}'(t-)$ *for all* $t \in \mathbb{R}$. *Then problem* (2.1) *admits a traveling wavefront.*

3. Existence of traveling wavefronts

Assume that $2\beta e^{-\gamma\tau} > \delta + \beta$, then we can get two equilibriums of (1.5):

$$k_1 = 0, \qquad k_2 = \left[\frac{2\beta e^{-\gamma\tau} - (\delta+\beta)}{\delta}\right]^{1/n} > 0.$$

We will show the existence of solutions of (3.1) with the asymptotic boundary conditions $\lim_{t\to-\infty}\phi(t) = k_1$ and $\lim_{t\to+\infty}\phi(t) = k_2$, which corresponds traveling wavefronts of (1.5) connecting k_1 and k_2.

Substituting $u(t,x) = \phi(s)$ into (1.5), and denoting the moving variable s still by t, the corresponding wave equation becomes

$$c\phi'(t) - \phi''(t) = -\delta\phi(t) - \frac{\beta\phi(t)}{1+\phi^n(t)} + \frac{2\beta e^{-\gamma\tau}\phi(t-c\tau)}{1+\phi^n(t-c\tau)}. \tag{3.1}$$

Define the function

$$f_c(\phi) = -\delta\phi(0) - \frac{\beta\phi(0)}{1+\phi^n(0)} + \frac{2\beta e^{-\gamma\tau}\phi(c\tau)}{1+\phi^n(c\tau)}.$$

Lemma 3.1. *If* $2\beta e^{-\gamma\tau} > \delta + \beta$, *then* $f_c(\hat{0}) = f_c(\hat{k_2}) = 0$, *and* $f_c(\hat{K}) \neq 0$, *for any* $K \in (k_1, k_2)$, *where* $\hat{K}$ *denotes the constant function taking the value* K *on* $[-c\tau, 0]$.

Next we show that $f_c(\phi)$ satisfies quasi-monotonicity condition by some assumption.

Lemma 3.2. *If* $2\beta e^{-\gamma\tau} > \delta + \beta$ *and* $\frac{2\beta e^{-\gamma\tau}-(\delta+\beta)}{\delta} < \frac{1}{n-1}$ *for* $n \in (1,\infty)$, *then* f_c *satisfies the following quasi-monotonicity condition:*

$$f_c(\phi) - f_c(\psi) + \alpha[\phi(0) - \psi(0)] \geqslant 0$$

for all $\phi, \psi \in C([-c\tau, 0], \mathbb{R})$ *with* $k_1 \leqslant \psi(s) \leqslant \phi(s) \leqslant k_2$, $s \in [-c\tau, 0]$, *where* $\alpha \geqslant \delta + \beta$.

Proof. Consider the function $h(y) = \frac{y}{1+y^n}$, it is clear that

$$h'(y) = \frac{1+(1-n)y^n}{(1+y^n)^2} \geqslant 0 \quad \text{for } 0 \leqslant y \leqslant \left(\frac{1}{n-1}\right)^{\frac{1}{n}}.$$

If $2\beta e^{-\gamma\tau} > \delta + \beta$ and $\frac{2\beta e^{-\gamma\tau}-(\delta+\beta)}{\delta} < \frac{1}{n-1}$, then

$$0 \leqslant \phi(t) \leqslant \psi(t) \leqslant k_2 = \left[\frac{2\beta e^{-\gamma\tau} - (\delta+\beta)}{\delta}\right]^{\frac{1}{n}} \leqslant \left(\frac{1}{n-1}\right)^{\frac{1}{n}}$$

since $n > 1$. It shows that the function $h(y)$ is increasing in $[k_1, k_2]$. A direct computation yields that

$$\begin{aligned} f_c(\phi) - f_c(\psi) &= -\delta\phi(0) - \frac{\beta\phi(0)}{1+\phi^n(0)} + \frac{2\beta\phi(c\tau)}{1+\phi^n(c\tau)}e^{-\gamma\tau} + \delta\psi(0) + \frac{\beta\psi(0)}{1+\psi^n(0)} - \frac{2\beta\psi(c\tau)}{1+\psi^n(c\tau)}e^{-\gamma\tau} \\ &= -\delta(\phi(0) - \psi(0)) - \beta\left(\frac{\phi(0)}{1+\phi^n(0)} - \frac{\psi(0)}{1+\psi^n(0)}\right) + 2\beta e^{-\gamma\tau}\left(\frac{\phi(c\tau)}{1+\phi^n(c\tau)} - \frac{\psi(c\tau)}{1+\psi^n(c\tau)}\right) \\ &\geqslant \left(-\delta - \frac{\beta}{1+\psi^n(0)}\right)(\phi(0) - \psi(0)) \geqslant (-\delta - \beta)(\phi(0) - \psi(0)) \end{aligned}$$

and if we choose $\alpha \geqslant \delta + \beta$, then we have

$$f_c(\phi) - f_c(\psi) + \alpha(\phi(0) - \psi(0)) \geqslant 0. \quad \square$$

Remark 3.1. If $n = 1$, the function $h(y) = \frac{y}{1+y^n}$ is increasing for all $y > 0$. Therefore the condition (A2) holds provided that $2\beta e^{-\gamma\tau} > \delta + \beta$.

Define the profile set

$$\Gamma^* = \left\{ \phi \in C(\mathbb{R}, \mathbb{R}) : \begin{array}{l} \text{(i) } \phi \text{ is nondecreasing in } \mathbb{R}, \\ \text{(ii) } \lim_{t\to-\infty} \phi(t) = k_1, \quad \lim_{t\to\infty} \phi(t) = k_2, \end{array} \right\}.$$

Next we will consider the solution of (3.1) by using the method of upper and lower solutions which are defined as follows:

Definition 3.1. The piecewise smooth functions $\overline{\phi}$ and $\underline{\phi}$ in $C(\mathbb{R}, \mathbb{R})$ are called upper and lower solution of (3.1) if $\overline{\phi} \geqslant \underline{\phi}$ and if

$$c\overline{\phi}'(t) - \overline{\phi}''(t) \geqslant -\delta\overline{\phi}(t) - \frac{\beta\overline{\phi}(t)}{1+\overline{\phi}^n(t)} + \frac{2\beta\overline{\phi}(t-c\tau)}{1+\overline{\phi}^n(t-c\tau)} e^{-\gamma\tau} \text{in } \mathbb{R}$$

and $\underline{\phi}$ satisfies the above differential inequalities in reversed order.

Define $\Delta_c(\lambda) = \lambda^2 - c\lambda + (\beta - \delta)$, then we have the following lemmas.

Lemma 3.3. *For $\beta > \delta$, the following statements are true,*

(i) *if $c < 2\sqrt{\beta - \delta}$, then $\Delta_c(\lambda) = 0$ has no real roots;*
(ii) *if $c = 2\sqrt{\beta - \delta}$, then $\Delta_c(\lambda) = 0$ has precisely one double root;*
(iii) *if $c > 2\sqrt{\beta - \delta}$, then $\Delta_c(\lambda) = 0$ has exactly two positive real roots $0 < \lambda_1 < \lambda_2$,*

$$\lambda_1 = \frac{c - \sqrt{c^2 - 4(\beta - \delta)}}{2}, \qquad \lambda_2 = \frac{c + \sqrt{c^2 - 4(\beta - \delta)}}{2},$$

and $\Delta_c(\lambda) < 0$ for all $\lambda \in (\lambda_1, \lambda_2)$.

Now we construct the upper solution of (3.1).

Lemma 3.4. *Assume $c > 2\sqrt{\beta - \delta}$ and $\tau \geq 0$ is sufficiently small. Then $\overline{\phi}(t) = \min\{k_2, e^{\lambda_1 t}\}$ is an upper solution of (3.1) and $\overline{\phi} \in \Gamma^*$.*

Proof. It is easy to show that $\overline{\phi} \in \Gamma^*$, so we omit it. Next we show that $\overline{\phi}$ is an upper solution of (3.1).

It needs to point out that if $\tau \geqslant 0$ is small, we have $\beta > \delta$ from the condition $2\beta e^{-\gamma\tau} > \delta + \beta$. Let t_0 be such that $e^{\lambda_1 t_0} = k_2$.

(i) For $t \geqslant t_0$, $\overline{\phi}(t) = k_2$, $\overline{\phi}'(t) = 0$, $\overline{\phi}''(t) = 0$, $\overline{\phi}(t - c\tau) \leqslant k_2$, thus

$$\begin{aligned} \overline{\phi}''(t) - c\overline{\phi}'(t) - \delta\overline{\phi}(t) - \frac{\beta\overline{\phi}(t)}{1+\overline{\phi}^n(t)} + \frac{2\beta\overline{\phi}(t-c\tau)}{1+\overline{\phi}^n(t-c\tau)} e^{-\gamma\tau} &\leqslant -\delta k_2 - \frac{\beta k_2}{1+k_2^n} + \frac{2\beta k_2}{1+k_2^n} e^{-\gamma\tau} \\ &= \frac{(-\delta - \delta k_2^n - \beta + 2\beta e^{-\gamma\tau})k_2}{1+k_2^n} = 0. \end{aligned}$$

(ii) For $t < t_0$, $\overline{\phi}(t) = e^{\lambda_1 t}$ and $\overline{\phi}(t - c\tau) = e^{\lambda_1(t - c\tau)}$, thus

$$\begin{aligned} &\overline{\phi}''(t) - c\overline{\phi}'(t) - \delta\overline{\phi}(t) - \frac{\beta\overline{\phi}(t)}{1+\overline{\phi}^n(t)} + \frac{2\beta\overline{\phi}(t-c\tau)}{1+\overline{\phi}^n(t-c\tau)} e^{-\gamma\tau} \\ &\leqslant \overline{\phi}''(t) - c\overline{\phi}'(t) - \delta\overline{\phi}(t) - \frac{\beta\overline{\phi}(t-c\tau)}{1+\overline{\phi}^n(t-c\tau)} + \frac{2\beta\overline{\phi}(t-c\tau)}{1+\overline{\phi}^n(t-c\tau)} e^{-\gamma\tau} \\ &\leqslant e^{\lambda_1 t}(\lambda_1^2 - c\lambda_1 - \delta - (\beta - 2\beta e^{-\gamma\tau})e^{-c\lambda_1\tau}) \\ &\leqslant e^{\lambda_1 t}[\lambda_1^2 - c\lambda_1 + (\beta - \delta)] = 0 \end{aligned}$$

since $\tau \geqslant 0$ is sufficiently small. According to the discussion above, we know that $\overline{\phi}$ is an upper solution of (3.1). □

Now let $c > 2\sqrt{\beta - \delta}$ and $0 < \lambda_1 < \lambda_2$ be the same as those given in Lemma 3.3. Take $\varepsilon > 0$ such that $\varepsilon < \lambda_1$ and $\lambda_1 + \varepsilon < \lambda_2$. Define $\underline{\phi} = \max\{0, (1 - Me^{\varepsilon t})e^{\lambda_1 t}\}$, where the constant $M > 1$ is to be determined.

Lemma 3.5. *If $\tau \geqslant 0$ is sufficiently small, for $M > \max\left\{1, \frac{\beta-\delta}{-\Delta_c(\lambda_1+\varepsilon)}\right\}$, $\underline{\phi}(t) = \max\{0, (1 - Me^{\varepsilon t})e^{\lambda_1 t}\}$ is a lower solution for Eq. (3.1).*

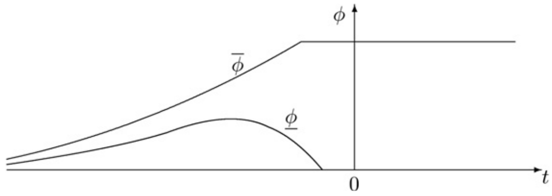

Fig. 1. The upper solution $\overline{\phi}$ and the lower solution $\underline{\phi}$.

Proof. Let $t_1 = \frac{1}{\varepsilon}\ln(\frac{1}{M})$, then $t_1 < 0$ for $M > 1$ and

$$\underline{\phi}(t) = \begin{cases} 0, & \text{for } t > t_1, \\ (1 - M\mathrm{e}^{\varepsilon t})\mathrm{e}^{\lambda_1 t}, & \text{for } t < t_1. \end{cases}$$

(i) For $t > t_1 + c\tau$, $\underline{\phi}(t) = 0$, $\underline{\phi}(t - c\tau) = 0$, $\underline{\phi}'(t) = 0$, and $\underline{\phi}''(t) = 0$. Hence

$$c\underline{\phi}'(t) - \underline{\phi}''(t) + \delta\underline{\phi}(t) + \frac{\beta\underline{\phi}(t)}{1 + \underline{\phi}^n(t)} - \frac{2\beta\underline{\phi}(t - c\tau)}{1 + \underline{\phi}^n(t - c\tau)}\mathrm{e}^{-\gamma\tau} = 0.$$

(ii) For $t_1 < t < t_1 + c\tau$, $\underline{\phi}(t) = 0$, $\underline{\phi}(t - c\tau) = (1 - M\mathrm{e}^{\varepsilon(t - c\tau)})\mathrm{e}^{\lambda_1(t - c\tau)} > 0$, $\underline{\phi}'(t) = 0$, and $\underline{\phi}''(t) = 0$. Hence

$$c\underline{\phi}'(t) - \underline{\phi}''(t) + \delta\underline{\phi}(t) + \frac{\beta\underline{\phi}(t)}{1 + \underline{\phi}^n(t)} - \frac{2\beta\underline{\phi}(t - c\tau)}{1 + \underline{\phi}^n(t - c\tau)}\mathrm{e}^{-\gamma\tau} < 0.$$

(iii) For $t < t_1$, we have $\underline{\phi}'(t) = [\lambda_1 - M(\varepsilon + \lambda_1)\mathrm{e}^{\varepsilon t}]\mathrm{e}^{\lambda_1 t}$, $\underline{\phi}''(t) = [\lambda_1^2 - M(\varepsilon + \lambda_1)^2\mathrm{e}^{\varepsilon t}]\mathrm{e}^{\lambda_1 t}$, $\underline{\phi}(t - c\tau) = [1 - M\mathrm{e}^{\varepsilon(t - c\tau)}]\mathrm{e}^{\lambda_1(t - c\tau)}$. It is easy to check that $0 \leqslant (1 - M\mathrm{e}^{\varepsilon(t - c\tau)})\mathrm{e}^{\lambda_1(t - c\tau)} < 1$. It follows that $\frac{1}{1 + \underline{\phi}^n(t - c\tau)} > 1 - \underline{\phi}(t - c\tau)$. Therefore

$$\begin{aligned} &\underline{\phi}''(t) - c\underline{\phi}'(t) - \delta\underline{\phi}(t) - \frac{\beta\underline{\phi}(t)}{1 + \underline{\phi}^n(t)} + \frac{2\beta\underline{\phi}(t - c\tau)}{1 + \underline{\phi}^n(t - c\tau)}\mathrm{e}^{-\gamma\tau} \\ &\quad \geqslant \underline{\phi}''(t) - c\underline{\phi}'(t) - \delta\underline{\phi}(t) - \frac{\beta\underline{\phi}(t)}{1 + \underline{\phi}^n(t)} + 2\beta\underline{\phi}(t - c\tau)(1 - \underline{\phi}(t - c\tau))\mathrm{e}^{-\gamma\tau} \\ &\quad \geqslant \underline{\phi}''(t) - c\underline{\phi}'(t) - (\delta + \beta)\underline{\phi}(t) + 2\beta\underline{\phi}(t - c\tau)(1 - \underline{\phi}(t - c\tau))\mathrm{e}^{-\gamma\tau} \\ &\quad = [\lambda_1^2 - M(\varepsilon + \lambda_1)^2\mathrm{e}^{\varepsilon t}]\mathrm{e}^{\lambda_1 t} - c[\lambda_1 - M(\varepsilon + \lambda_1)\mathrm{e}^{\varepsilon t}]\mathrm{e}^{\lambda_1 t} - (\delta + \beta)(1 - M\mathrm{e}^{\varepsilon t})\mathrm{e}^{\lambda_1 t} \\ &\qquad + 2\beta(1 - M\mathrm{e}^{\varepsilon(t - c\tau)})\mathrm{e}^{\lambda_1(t - c\tau)}[1 - (1 - M\mathrm{e}^{\varepsilon(t - c\tau)})\mathrm{e}^{\lambda_1(t - c\tau)}]\mathrm{e}^{-\gamma\tau}. \end{aligned}$$

If $\tau > 0$ is sufficiently small, we obtain

$$\begin{aligned} &\underline{\phi}''(t) - c\underline{\phi}'(t) - \delta\underline{\phi}(t) - \frac{\beta\underline{\phi}(t)}{1 + \underline{\phi}^n(t)} + \frac{2\beta\underline{\phi}(t - c\tau)}{1 + \underline{\phi}^n(t - c\tau)}\mathrm{e}^{-\gamma\tau} \\ &\quad \geqslant \mathrm{e}^{\lambda_1 t}\left[\Delta_c(\lambda_1) - M\mathrm{e}^{\varepsilon t}\Delta_c(\lambda_1 + \varepsilon) - \mathrm{e}^{\lambda_1 t}(1 - M\mathrm{e}^{\varepsilon t})^2(\beta - \delta)\right] \\ &\quad \geqslant \mathrm{e}^{\lambda_1 t}[-M\mathrm{e}^{\varepsilon t}\Delta_c(\lambda_1 + \varepsilon) - \mathrm{e}^{\varepsilon t}(1 - M\mathrm{e}^{\varepsilon t})^2(\beta - \delta)] \\ &\quad = \mathrm{e}^{(\lambda_1 + \varepsilon)t}[-M\Delta_c(\lambda_1 + \varepsilon) - (1 - M\mathrm{e}^{\varepsilon t})^2(\beta - \delta)], \end{aligned}$$

since $t < t_1 < 0$ and $\varepsilon < \lambda_1$.

Note that $0 \leqslant 1 - M\mathrm{e}^{\varepsilon t} < 1$, when $t < t_1 < 0$. So we have $(1 - M\mathrm{e}^{\varepsilon t})^2 < 1$. Thus

$$\begin{aligned} &\underline{\phi}''(t) - c\underline{\phi}'(t) - \delta\underline{\phi}(t) - \frac{\beta\underline{\phi}(t)}{1 + \underline{\phi}^n(t)} + \frac{2\beta\underline{\phi}(t - c\tau)}{1 + \underline{\phi}^n(t - c\tau)}\mathrm{e}^{-\gamma\tau} \\ &\quad \geqslant \mathrm{e}^{(\lambda_1 + \varepsilon)t}[-\Delta_c(\lambda_1 + \varepsilon)]\left[M - \frac{\beta - \delta}{-\Delta_c(\lambda_1 + \varepsilon)}\right]. \end{aligned}$$

If we choose $M > \max\left\{1, \frac{\beta - \delta}{-\Delta_c(\lambda_1 + \varepsilon)}\right\}$, thus $\underline{\phi}$ is a lower solution of (3.1). □

The graph of $\underline{\phi}$ together with the upper solution $\overline{\phi}$ is shown in Fig. 1.

It is clear that $k_1 \leqslant \underline{\phi} \leqslant \overline{\phi} \leqslant k_2$. To summarize the above conclusions we give our main result of this paper below.

Theorem 3.6. *If* $2\beta e^{-c\gamma\tau} > \delta+\beta$, $\frac{2\beta e^{-\gamma\tau}-(\delta+\beta)}{\delta} < \frac{1}{n-1}$ *for* $n \in (1, \infty)$, *and* $\tau \geqslant 0$ *is small enough, then for every* $c > 2\sqrt{\beta-\delta}$, *problem (1.5) has a traveling wavefront which connects the equilibriums* $k_1 = 0$ *and* $k_2 = \left[\frac{2\beta e^{-\gamma\tau}-(\delta+\beta)}{\delta}\right]^{\frac{1}{n}}$.

References

[1] M.C. Mackey, Unified hypothesis for the origin of aplastic anemia and periodic hematopoiesis, Blood 51 (1978) 941–956.
[2] M.C. Mackey, L. Glass, Oscillation and chaos in physiological control system, Science 197 (1977) 287–289.
[3] P.X. Weng, Z.P. Dai, Global atractivity for a model of hemapoiesis, J. South China Normal Univ. 2 (2001) 12–19.
[4] X. Wang, Z.X. Li, Dynamics for a type of general reaction–diffusion model, Nonlinear Anal.: TMA 67 (2007) 2699–2711.
[5] K. Gopalsamy, M.R. Kulenvic, G. Ladas, Oscillation and global attrativity in model of mematopoiesis, J. Dynam. Differential Equations 22 (1990) 117–132.
[6] S.R. Dunbar, Travelling wave solutions of diffusive Lotka–Volterra equations, J. Math. Biol. 17 (1983) 11–32.
[7] R. Gardner, Existence and stability of travelling wave solutions of predator–prey systems via the connection index, SIAM J. Appl. Math. 44 (1984) 56–79.
[8] J. Huang, X. Zou, Existence of travelling wave fronts of delayed reaction diusion systems without monotonicity, Discrete Contin. Dyn. Syst. 9 (2003) 925–936.
[9] G. Lin, W.T. Li, Travelling wavefronts of Belousov–Zhabotinskii system with diffusion and delay, Appl. Math. Lett. 22 (2009) 341–346.
[10] S. Ma, Travelling wavefronts for delayed reaction–difusion systems via a fixed point theorem, J. Differential Equations 171 (2001) 294–314.
[11] Y.H. Peng, Y.L. Song, Existence of traveling wave solutions for a reaction–diffusion equation with distributed delays, Nonlinear Anal.: TMA 67 (2007) 2415–2423.
[12] K. Schaaf, Asymptotic behavior and travelling wave solutions for parabolic functional differential equations, Trans. Amer. Math. Soc. 302 (1987) 587–615.
[13] S.L. Wu, S.Y. Liu, Uniqueness of non-monotone traveling waves for delayed reaction–diffusion equations, Appl. Math. Lett. (2009) doi:10.1016/j.aml.2009.01.014.
[14] J. Wu, X. Zou, Traveling wave fronts of reaction–diffusion systems, J. Dynam. Differential Equations 13 (2001) 651–687.